Instability mechanism at driven contact lines

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An explanation of the mechanism for the fingering instability at driven contact lines is presented. Semiquantitative predictions for the growth of the fingers as a function of time, the most unstable wavelength, and the initial growth rate are deduced. These predictions are consistent with recent experiments of de Bruyn [Phys. Rev. A **46**, R4500 (1992)].

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Driven contact lines appear in numerous applications where a solid surface is coated with a thin layer of viscous fluid [1]. Experiments on such systems have shown [2–6] that when a contact line is driven by a constant force, the fluid interface is unstable and breaks up into fingers. The instability presents a practical problem as it disrupts the uniform coating of the solid surface. It also raises theoretical issues involving the dynamics of contact lines, for the shape of the fingers depends on the fluid characteristics.

The instability has been studied experimentally for flow down an inclined plane [2,3,5,6] and flow on a rotating table [4]. Theoretical efforts have mainly focused on characterizing the base state of the flow, before the instability. Huppert [2] characterized the outer region, far from the contact line. The inner region was first described correctly by Troian *et al.* [7] and by Hocking [8]; more rigorous descriptions have since been given by Moriarty, Schwartz, and Tuck [9] and by Goodwin and Homsy [10]. The instability of the base state in the gravitational case has been addressed by Troian *et al.* [7], who suggest that it is linearly unstable. However, their theory applies in a parameter regime outside the range of recent experiments. No theoretical discussion of late-time-flow characteristics of the fingers has been given.

In this paper, I present a physical picture of the mechanism that causes the instability. The argument suggests that the growth of the fingers is driven by macroscopic flows, and is not significantly influenced by contact-line dynamics. Moreover, the picture predicts many characteristics of the flow. Focusing on the gravity-driven case, I extend the results of Troian *et al.* [7] to the parameter range of recent experiments. At early times, the finger length grows exponentially, and at later times, the growth is linear. The most unstable wavelength and the initial growth rate of the perturbation are also predicted. The results are consistent with recent experiments of de Bruyn [6].

The geometry of the problem is depicted in Fig. 1. The hydrodynamic equation for the height $h(x,y,t)$ of the fluid is [11]

$$3\mu h_t+\nabla\cdot[\gamma h^3\nabla(\nabla^2 h)-\rho g\cos(\alpha)h^3\nabla h+\rho g\sin(\alpha)h^3\hat{\mathbf{x}}]=0\ . \quad (1)$$

The derivation of the equation uses a lubrication approximation of the Navier-Stokes equation, neglecting the inertial terms. These approximations are appropriate for thin films of viscous fluid. Here, μ is the viscosity of the fluid, ρ is the fluid density, γ is the surface tension, g is the gravitational acceleration, and α is the inclination angle of the plane. The fluid is initially arranged so that the contact line is straight. After the flow begins, the height profile develops two regions. The outer region, far from the contact line, is described by a similarity solution of (1) where the $\rho g\sin(\alpha)h^3$ term dominates [2]. The length of this region, x_N, increases like $t^{1/3}$, and the thickness decreases like $t^{-1/2}$. Volume conservation implies that the outer solution ends abruptly with a height H_N.

Near x_N, the edge of the outer region, the profile is smoothed by surface tension. Troian *et al.* [7] computed a solution in this *inner region* by assuming that the outer region is quasisteady, and then finding a solution moving at the velocity $U=\dot{x}_N$ [12]. The most important feature of their solution is that there is a characteristic "bump" near the contact line (see Fig. 1). Furthermore, their solution suggests the following scalings for flows in the inner region:

$$h=\bar{h}H_N\ ,$$
$$(x,y)=(\bar{x}l,\bar{y}l)\ , \quad (2)$$
$$t=\bar{t}(l/U)\ .$$

Here, $l=H_N/(3\mathrm{Ca})^{1/3}$, where $\mathrm{Ca}=\mu U/\gamma$ is the capillary number. When expressed in terms of these units, the height and width of the base profile remain nearly con-

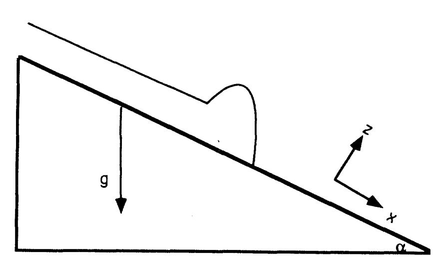

FIG. 1. Schematic diagram of the apparatus. The y axis points into the page.

stant in time. With these rescalings (dropping the bars), Eq. (1) becomes [13]

$$h_t + h^2 h_x + \nabla \cdot [h^3 \nabla \nabla^2 h - \cot(\alpha)(3\mathrm{Ca})^{1/3} h^3 \nabla h] = 0 \ . \quad (3)$$

It is useful for what follows to interpret the terms in Eq. (3): $h^2 h_x$ is a convective term and makes the velocity of the profile depend quadratically on its height. The two other terms are diffusive and tend to flatten the profile. The inner region is a consequence of the competition between these two types of terms, as the convective term tends to form a shock, and the diffusive terms smooth it out. The "bump" is a consequence of the balance between curvature gradients and viscous stresses.

Now I consider the instability of this profile. Imagine an interface with a shape initially independent of the y direction, perturbed by a sinusoidal perturbation of wavelength λ. The equiheight lines of such a perturbation are shown in Fig. 2. The bold line represents the maximum height of the profile. The crucial point is that under such a perturbation, the diffusive terms cause fluid to flow in the direction transverse to the main flow [14], decreasing the height of the fluid at positions A and C, the troughs [15]. The height at position A (C) is then less than the height at position B, the tip. Since the velocity increases like h^2, the tip (B) then moves faster than the troughs (A,C), resulting in the formation of a finger. It is important to note that these transverse flows will only be induced for sufficiently long wavelength perturbations, for the $h^3 \nabla \nabla^2 h$ term also tends to decrease the additional curvature of the interface in the x-y plane, induced by the perturbation.

This picture of the instability suggests a reason for the difference in finger shapes between partial-wetting and complete-wetting fluids. A partial-wetting fluid has a finite equilibrium contact angle. When the height of the trough decreases enough so that the contact angle is equal to its equilibrium value, the velocity of the trough must be zero. If on the other hand, the equilibrium contact angle is zero, then the tip and trough can move down the plane together; in this case, Eq. (3) should capture the essential dynamics of the fluid. The dependence of finger shape on contact angle has been previously noted by Silvi and Dussan [3] and Jerrett and de Bruyn [5].

Several semiquantitative consequences can be drawn from this picture of the instability. First, I consider the growth of the finger with time. As emphasized by Troian *et al.* [7], it is proper to consider the initial development of the instability in the frame of reference of Eq. (3). There are two regimes: Shortly after the onset of the instability, the growth of the finger is dominated by the flux of fluid, Φ, from the troughs (A,C in Fig. 3) to the tip (B in Fig. 3). A superposition of the flow profiles at A and B in this regime is shown in Fig. 3. The mass M of the finger has the time dependence $\dot{M} \sim \Phi L(t)$, where $L(t)$ is the finger length. Since the height is 1 in units of Eq. (3), $M \sim \lambda L(t)$, and thus $\dot{L} \sim \beta L$, so that the early time growth of the finger is exponential. This regime will end when $L(t)$ is of order of the thickness W of the "bump" [16]. At later times, the height of the fluid at the tip and the trough of the finger remains constant, with the tip higher than the trough. Since the velocity of the flow varies like h^2, there is a constant velocity difference between the tip and the trough. Thus, the finger length increases linearly in time. These time dependences of the finger length were measured by de Bruyn [6]. In his data, the transition between the two regimes occurs around 5–10 (in units of l), depending on the inclination angle. de Bruyn does not report the thickness of the "bump" in his experiments; however, the computed solution of Goodwin and Homsy with $\alpha = 45°$ [10] has a "bump" with $W \sim 6$. The variation in de Bruyn's transition lengths with inclination angle is probably due to the fact that, in general, W is a weak function of α when expressed in units of l.

Now I estimate the growth rate, and the most unstable wavelength of the pattern. From above, the growth rate of the fingers is $\beta(\lambda) \simeq \Phi/\lambda$. The flux Φ is the y component of the flux in Eq. (3). Dimensional analysis on this flux gives the estimate for $\beta(\lambda)$

$$\beta(\lambda) \simeq 2 \left[\frac{1}{W^2 \lambda(\lambda/2)} - \frac{1}{\lambda(\lambda/2)^3} + \frac{\cot(\alpha)(3\mathrm{Ca})^{1/3}}{(\lambda/2)\lambda} \right] . \quad (4)$$

The most unstable mode λ^* is obtained by maximizing $\beta(\lambda)$. This gives

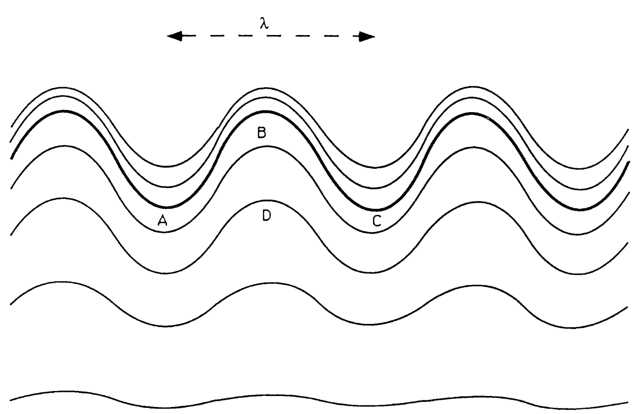

FIG. 2. Equiheight lines of the perturbed interface. A and C are trough positions, and B is the position of a tip. The bold line represents the maximum height of the profile. The uppermost line represents the contact line. Fluid flows "downhill," from points A and C to point D.

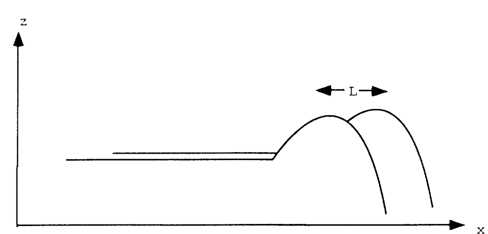

FIG. 3. Superposition of tip and trough profiles at early times.

$$(\lambda^*)^2 = \frac{8W^2}{[1+\cot(\alpha)(3\mathrm{Ca})^{1/3}W^2]} . \quad (5)$$

As above, W is about 6; in de Bruyn's experiments, $\cot(\alpha)(3\mathrm{Ca})^{1/3} \sim 1$, so that Eq. (5) gives $\lambda^* \sim 3$. Using (4), the growth rate is $\beta \sim 0.2$. These results are the same order of magnitude as the measurements of de Bruyn [6], who finds that $\lambda^* \sim 9$, and $\beta \sim 0.1$.

These results demonstrate that the simple physics proposed here lead to reasonable estimates for the measured quantities. Furthermore, the results show that the measured quantities λ^* and β depend on the dimensionless parameter $(3\mathrm{Ca})^{1/3}\cot(\alpha)$ neglected by Troian *et al.* [7]. At large angles where $(3\mathrm{Ca})^{1/3}\cot(\alpha)$ *is* negligible, I recover Troian *et al.*'s result that the most unstable wavelength is constant in units of l. This is true because at large angles, W becomes independent of α. However, at the small angles accessible to experiments, the measured quantities depend on the inclination angle, *even when expressed in units of l.* In fact, (5) implies that the dimensionless wavelength of the pattern λ^* decreases, and the dimensionless growth rate β increases, at small values of α [17]. Jerrett and de Bruyn [5] have shown experimentally that the wavelength decreases as a function of α for partially wetting fluids. de Bruyn's experiments [6] on silicone oil also seem to suggest that λ^* decreases at small angles. Fewer experiments have measured β at small angles. de Bruyn's experiments seem to suggest that β actually decreases at small angles. More work on this point is necessary.

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[13] Troian *et al.* neglect the third flux term in their analysis of the equation. This approximation is valid as long as $\cot(\alpha)(3\mathrm{Ca})^{1/3} \ll 1$. This condition is not met in de Bruyn's experiment.
[14] The fact that the instability is initially driven by transverse flows was first noted in the computer simulations of L. Schwartz, Phys. Fluids A **1**, 444 (1989).
[15] Here "trough" and "tip" refer to finger characteristics, *not* to the height of the profile.
[16] The "thickness" W of the "bump" refers to the extent of the "bump" in the x direction.
[17] Note, however, that if the inclination angle becomes too small, the width of the inner region becomes of the order of the size of the system so that the arguments of this paper are invalid.